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Research Article

Nevanlinna–Pick Interpolation in Selected Subalgebras of $H^\infty(\mathbb{D})$

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Abstract

Given a collection K of positive integers, let $H^\infty_K(\mathbb{D})$ denote the set of all bounded analytic functions defined on the unit disk $\mathbb{D}$ in $\mathbb{C}$ whose k^{th} derivative vanishes at zero, for all $k \in K$. We establish a Nevanlinna-Pick interpolation result for the subalgebra $H^\infty_K(\mathbb{D})$. This work is a generalization of the interpolation theorem that Davidson, Paulsen, Raghupathi, and Singh proved for the algebra $H^\infty_{\{1\}}(\mathbb{D})$ in [9].

1 Introduction

The algebras we are interested in this paper are defined as follows:
Let $K \subset \mathbb{Z}_+$ and define

$$H^\infty_K(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f^{(k)}(0) = 0, \quad \text{for} \quad \text{all} \ \ k \in K\}.$$

We consider those sets K for which $H^\infty_K(\mathbb{D})$ is an algebra under pointwise multiplication of functions. Obviously, not every set K defines an algebra (for example, let $K = \{2\}$). Though there is not a complete characterization of the set K for which $H^\infty_K(\mathbb{D})$ is an algebra, Ryle and Trent [20] have given certain criteria that the set K must meet. We state some of these criteria as Lemmas 3.1 and 3.2 in Section 3.

Furthermore, we endow $H^\infty(\mathbb{D})$ with the usual supremum norm, which we define below.

Definition 1.1. Let f be a function defined on $\mathbb{D}$. The *supremum norm* $||f||_\infty$ of f is defined as the least upper bound for the set of all moduli of f. Formally, we write

$$||f||_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

Nevanlinna and Pick proved an interpolation result for the algebra $H^\infty(\mathbb{D})$. Their theorem shows that an interpolant exists in this algebra precisely when the associated Pick matrix is positive semidefinite.

Nevanlinna–Pick Interpolation Theorem. *Given distinct points $z_1, z_2, ..., z_n$ in the unit disk $\mathbb{D}$ in the complex plane and complex numbers $w_1, w_2, ..., w_n$ in $\mathbb{D}$, there exists a function f in $H^\infty(\mathbb{D})$ with $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$ and $\|f\|_\infty \leq 1$ if and only if the associated Pick matrix*

$$\left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{i,j} \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

Davidson, Paulsen, Raghupathi, and Singh [9] proved an interpolation result for $H^\infty_{\{1\}}(\mathbb{D})$, the collection of all bounded, analytic functions f on $\mathbb{D}$, satisfying the constraint $f'(0) = 0$ (see Theorem 1.2 in [9]).

Interpolation Theorem on $H^\infty_{\{1\}}(\mathbb{D})$ (Davidson, Paulsen, Raghupathi, and Singh). *Let $z_1, z_2, ..., z_n$ be distinct points in $\mathbb{D}$, and let $w_1, w_2, ..., w_n$, be in $\mathbb{D}$. Then there exists an analytic function f in $H^\infty_{\{1\}}(\mathbb{D})$ with $\|f\|_\infty \leq 1$ such that $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$ if and only if there exists $\lambda \in \mathbb{D}$ so that*

$$\left[\frac{z_i^2\overline{z_j^2} - \varphi_\lambda(w_i)\overline{\varphi_\lambda(w_j)}}{1 - z_i\overline{z_j}}\right]_{i,j} \quad \text{is} \quad \text{positive} \quad \text{semidefinite,}$$

where $\varphi_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda}z}$ denotes the elementary Möbius transformation.

Observing that all functions in $H^\infty_{\{1\}}(\mathbb{D})$ have first derivatives that vanish at zero, it is natural to define our desired set as follows.

Definition 1.2. Let K denote a nonempty collection of positive integers and define the set $H^\infty_K(\mathbb{D})$ as the set of all functions in $H^\infty(\mathbb{D})$ whose k^{th} derivative vanishes at zero, for all $k \in K$. Formally, define

$$H^\infty_K(\mathbb{D}) := \{f \in H^\infty(\mathbb{D}) : f^{(k)}(0) = 0, \text{ for all } k \in K\}.$$

We assume that $H^\infty_K(\mathbb{D})$ is an algebra. As we shall see in Section 2, this assumption is necessary.

Although [9] is the primary inspiration for this work, Carleson's corona theorem and Wolff's problem of ideals have also been established for $H^\infty_K(\mathbb{D})$, see [18] and [7], respectively. Thus, we found it plausible that a similar interpolation condition holds for the subalgebra $H^\infty_K(\mathbb{D})$. Indeed, we will prove the interpolation results for $H^\infty_K(\mathbb{D})$ in Theorem 4.1 and Corollary 4.2 in Section 4 of this paper.

In the next section, we will briefly consider examples of sets K that may or may not yield an algebra, demonstrating that we must make the assumption that $H^\infty_K(\mathbb{D})$ is an algebra. Section 3 is devoted to all results that will be used to assist us in proving Theorem 4.1 and Corollary 4.2.

2 The Subalgebra $H^\infty_K(\mathbb{D})$

We consider those sets K for which $H^\infty_K(\mathbb{D})$ is an algebra. As the next example shows, not all sets K yield an algebra.

Example 2.1. The set $H^\infty_{\{2\}}(\mathbb{D})$ is not an algebra, because it fails to be closed under pointwise multiplication. Consider the functions $f(z) = z$ and $g(z) = z$ on $\mathbb{D}$. Clearly, f and g are analytic and bounded, so $f, g \in H^\infty(\mathbb{D})$. Further, $f''(0) = g''(0) = 0$, so $f, g \in H^\infty_{\{2\}}(\mathbb{D})$. But for the product $f(z)g(z) = z^2$, we have $\big(fg\big)''(z) = 2$, which does not vanish at zero. Thus, $fg \notin H^\infty_{\{2\}}(\mathbb{D})$, so it is not closed under pointwise multiplication. Therefore, $H^\infty_{\{2\}}(\mathbb{D})$ is not an algebra.

However, the set $H^\infty_{\{1,3\}}(\mathbb{D})$ is an algebra.

Example 2.2. Let $f, g \in H^\infty_{\{1,3\}}(\mathbb{D})$. For the sum $f + g$, we have

$$(f+g)'(0) = f'(0) + g'(0) = 0 \quad \text{and} \quad (f+g)'''(0) = f'''(0) + g'''(0) = 0,$$

so $f + g \in H^\infty_{\{1,3\}}(\mathbb{D})$. For the first derivative of the product fg,

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z),$$

which implies that

$$\begin{aligned}(fg)'(0) &= f'(0)g(0) + f(0)g'(0)\\ &= 0 \cdot g(0) + f(0) \cdot 0\\ &= 0.\end{aligned}$$

Similarly, the third derivative of fg yields

$$(fg)'''(z) = f'''(z)g(z) + 3f''(z)g'(z) + 3f'(z)g''(z) + f(z)g'''(z),$$

and thus

$$\begin{aligned}(fg)'''(0) &= f'''(0)g(0) + 3f''(0)g'(0) + 3f'(0)g''(0) + f(0)g'''(0)\\ &= 0 \cdot g(0) + 3f''(0) \cdot 0 + 3 \cdot 0 \cdot g''(0) + f(0) \cdot 0\\ &= 0.\end{aligned}$$

Therefore, $fg \in H^\infty_{\{1,3\}}(\mathbb{D})$. Since $H^\infty_{\{1,3\}}(\mathbb{D})$ is closed under both pointwise multiplication and addition, we conclude that it is an algebra.

The above examples show that not every K yields an algebra. A complete characterization of such sets K is an open problem worthy of investigation. Nevertheless, we will proceed to assume that a K is given such that $H^\infty_K(\mathbb{D})$ is an algebra.

3 Preliminaries

Ryle and Trent have conducted a thorough analysis on the subalgebra $H^\infty_K(\mathbb{D})$, revealing the rich structure and interesting properties it possesses (see Chapter 4 in [20]). Furthermore, they provide necessary criteria that K must satisfy

whenever $H^\infty_K(\mathbb{D})$ is an algebra. We catalog some of these attributes below (see Lemmas 4.2.1, 4.2.2, 4.2.4, and Corollary 4.2.1 in [20]).

Lemma 3.1. ***(Ryle and Trent)*** *Let $K \subset \mathbb{Z}_+$ such that $H^\infty_K(\mathbb{D})$ is an algebra. Then*

(i) $k_0 \notin K$ if and only if $\varphi(z) = z^{k_0} \in H^\infty_K(\mathbb{D})$.
(ii) If $j, k \notin K$, then $j + k \notin K$.
(iii) If $j \notin K$, then $nj \notin K$ for all $n \geq 2$.
(iv) Suppose $k_0 \in K$. If $1 < j < k_0$ satisfies $j \notin K$, then $k_0 - j \in K$.

Given an algebra $H^\infty_K(\mathbb{D})$, the following lemma demonstrates that some collection of positive integers absent from K exists. (see Corollary 1 in [19]).

Lemma 3.2. ***(Ryle and Trent)*** *If $H^\infty_K(\mathbb{D})$ is an algebra, then there exists $d \in \mathbb{Z}_+$, a finite set $\{n_i\}_{i=1}^p \subset \mathbb{Z}_+$ with $n_1 < \cdots < n_p$ and $\gcd(n_1, \ldots, n_p) = 1$, and a positive integer $N_0 > n_p$ so that*

$$\mathbb{Z}_+ \setminus K = \{n_1 d, n_2 d, \ldots, n_p d, N_0 d, (N_0 + j)d : j \in \mathbb{Z}_+\}.$$

These lemmas show that every function $f \in H^\infty_K(\mathbb{D})$ has the following power series representation

$$f(z) = \alpha_0 + \alpha_1 z^{n_1 d} + \alpha_2 z^{n_2 d} + \cdots + \alpha_p z^{n_p d} + \alpha_{p+1} z^{N_0 d} + \sum_{j=1}^{\infty} \alpha_{p+j+1} z^{(N_0+j)d},$$

where the coefficient $\alpha_i \in \mathbb{C}$. In order to improve our understanding of these results, we provide a couple of examples below.

Example 3.3. Given $H^\infty_K(\mathbb{D})$ we show the particular power series representation of functions in this algebra.

i. If $f \in H^\infty_{\{1\}}(\mathbb{D})$, then the power series representation of f is

$$f(z) = \alpha_0 + \alpha_2 z^2 + \alpha_3 z^3 + \alpha_4 z^4 + \sum_{i=5}^{\infty} \alpha_i z^i.$$

ii. If $f \in H^\infty_{\{1,3\}}(\mathbb{D})$, then the power series representation of f is

$$f(z) = \alpha_0 + \alpha_2 z^2 + \alpha_4 z^4 + \alpha_5 z^5 + \sum_{i=6}^{\infty} \alpha_i z^i.$$

Because we will be taking derivatives of function compositions, we will be using Bruno's Formula below (see [13] for more information).

Bruno's Formula. *Let f and g denote functions with a sufficient number of derivatives. Then the k^{th} derivative of the composition $g \circ f$ is*

$$\frac{d^k}{dz^k}\Big[g(f(z))\Big] = \sum \frac{k!}{b_1!\, b_2! \cdots b_k!} \cdot g^{(b)}(f(z)) \cdot \prod_{\ell=1}^{k} \Big(\frac{f^{(\ell)}(z)}{\ell!}\Big)^{b_\ell}$$

where the sum is over all k-tuples of nonnegative integers $(b_1, \ldots, b_k)$ satisfying the constraint

$$\sum_{\ell=1}^{k} \ell\, b_\ell = k,$$

and $b := b_1 + b_2 + \cdots + b_k$.

We use Bruno's formula in the following example, which serves as a motivation for Lemmas 3.5 and 3.6 below.

Example 3.4. When $k = 3$, Bruno's formula shows that we take the triples (b_1, b_2, b_3) satisfying the equation $b_1 + 2b_2 + 3b_3 = 3$. Clearly, the only solutions are $(3, 0, 0)$, $(1, 1, 0)$ and $(0, 0, 1)$. Given $(3, 0, 0)$, the first term has the form

$$\frac{3!}{3!0!0!} \cdot g^{(3+0+0)}(f(z)) \cdot \left(\frac{f^{(1)}(z)}{1!}\right)^3 \left(\frac{f^{(2)}(z)}{2!}\right)^0 \left(\frac{f^{(3)}(z)}{3!}\right)^0 = g^{(3)}(f(z))\big[f^{(1)}(z)\big]^3.$$

Further, the solution $(1,1,0)$ yields the term

$$\frac{3!}{1!1!0!}\cdot g^{(1+1+0)}(f(z))\cdot\left(\frac{f^{(1)}(z)}{1!}\right)^1\left(\frac{f^{(2)}(z)}{2!}\right)^1\left(\frac{f^{(3)}(z)}{3!}\right)^0 = 3g^{(2)}(f(z))f^{(1)}(z)f^{(2)}(z).$$

Finally, the solution $(0,0,1)$ provides the last term of the form

$$\frac{3!}{0!0!1!}\cdot g^{(0+0+1)}(f(z))\cdot\left(\frac{f^{(1)}(z)}{1!}\right)^0\left(\frac{f^{(2)}(z)}{2!}\right)^0\left(\frac{f^{(3)}(z)}{3!}\right)^1 = g^{(1)}(f(z))f^{(3)}(z).$$

Therefore,

$$(g\circ f)^{(3)}(z) = g^{(3)}(f(z))\big[f^{(1)}(z)\big]^3 + 3g^{(2)}(f(z))f^{(1)}(z)f^{(2)}(z) + g^{(1)}(f(z))f^{(3)}(z).$$

Note that each solution to the constraint generates a term in the above equation. Now, consider the algebra $H^\infty_{\{1,3\}}(\mathbb{D})$. If $f\in H^\infty_{\{1,3\}}(\mathbb{D})$, we see from the first three equations that a nonzero component b_j in (b_1,b_2,b_3) exists such that $j\in\{1,3\}$. We show that this is always true.

Lemma 3.5. *Suppose K is a set of positive integers such that $H^\infty_K(\mathbb{D})$ is an algebra. If $k\in K$, then for every k-tuple $(b_1,b_2,\dots,b_k)$ satisfying*

$$\sum_{\ell=1}^{k}\ell\, b_\ell = k,$$

there exists an $\ell_j\in\{\ell_1,\ell_2,\dots,\ell_t\}$ such that $\ell_j\in K$, where $b_{\ell_1},b_{\ell_2},\dots,b_{\ell_t}$ are the nonzero components of $(b_1,b_2,\dots,b_k)$.

Proof. Let $k\in K$. Assume instead that $\ell_j\notin K$, for all $\ell_j\in\{\ell_1,\ell_2,\dots,\ell_t\}$. By Lemma 3.1.(iii), the product $\ell_j\, b_{\ell_j}\notin K$. Further,

$$\sum_{j=1}^{t}\ell_j\, b_{\ell_j}\notin K,$$

by Lemma 3.1.(ii). Since $b_m=0$ for each $m\in\{1,2,\dots,k\}\setminus\{\ell_1,\ell_2,\dots,\ell_t\}$,

$$\sum_{\ell=1}^{k}\ell\, b_\ell = \sum_{j=1}^{t}\ell_j\, b_{\ell_j} = k\notin K,$$

which is a contradiction, as we know $k\in K$. □

We now show that $H^\infty_K(\mathbb{D})$ is a right ideal in $H^\infty(\mathbb{D})$.

Lemma 3.6. *Suppose that $H^\infty_K(\mathbb{D})$ is an algebra. If $g\in H^\infty(\mathbb{D})$ and $f\in H^\infty_K(\mathbb{D})$, then $g\circ f\in H^\infty_K(\mathbb{D})$.*

Proof. Let $g\in H^\infty(\mathbb{D})$ and $f\in H^\infty_K(\mathbb{D})$. Then $g\circ f\in H^\infty(\mathbb{D})$. If $k\in K$, Bruno's formula allows us to write

$$\big(g(f(z))\big)^{(k)} = \sum\frac{k!}{b_1!\,b_2!\cdots b_k!}\cdot g^{(b)}(f(z))\cdot\prod_{\ell=1}^{k}\Big(\frac{f^{(\ell)}(z)}{\ell!}\Big)^{b_\ell}$$

where the sum is over all k-tuples of nonnegative integers $(b_1,\dots,b_k)$ satisfying the constraint

$$\sum_{\ell=1}^{k}\ell\, b_\ell = k,$$

and $b:=b_1+b_2+\cdots+b_k$. For each solution $(b_1,b_2,\dots,b_k)$, there exist nonzero components $b_{\ell_1},b_{\ell_2},\dots,b_{\ell_t}$ in $(b_1,b_2,\dots,b_k)$, because $k>0$. By Lemma 3.5, there exists an $\ell_j\in\{\ell_1,\ell_2,\dots,\ell_t\}$ such that $\ell_j\in K$, and thus

$$\left(\frac{f^{(\ell_j)}(z)}{\ell_j!}\right)^{b_{\ell_j}} = 0 \quad\text{at}\quad z=0.$$

Therefore, every term in the sum vanishes at $z=0$, so $(g(f(0)))^{(k)}=0$. Hence, $g\circ f\in H^\infty_K(\mathbb{D})$. □

We now establish a trivial inequality.

Lemma 3.7. *Suppose $K = \{1, k_1, k_2, \ldots, k_p\}$ is a subset of $\mathbb{Z}_+$ where $k_p = \max K$. If m is the smallest positive integer not in K, then $k_p < m\left\lceil \frac{k_p}{m} \right\rceil$.*

Proof. Since

$$\left\lfloor \frac{k_p}{m} \right\rfloor \leq \frac{k_p}{m} < \left\lfloor \frac{k_p}{m} \right\rfloor + 1 = \left\lceil \frac{k_p}{m} \right\rceil$$

we conclude that

$$k_p < m\left\lceil \frac{k_p}{m} \right\rceil.$$

□

The following propositions will be used to prove Lemma 3.11 below. We first show that if $H^\infty_K(\mathbb{D})$ is an algebra, $1 \in K$.

Proposition 3.8. *If $K \subset \mathbb{Z}_+$ is nonempty such that $H^\infty_K(\mathbb{D})$ is an algebra, then $1 \in K$.*

Proof. Assume to the contrary that $1 \notin K$. Let k be the smallest positive integer in K. Then $k - 1 \notin K$ where $1 \leq k - 1$. If $k - 1 = 1$, then $k = 2 \in K$. By Lemma 3.1.(i), $z \in H^\infty_K(\mathbb{D})$ and $z^2 \notin H^\infty_K(\mathbb{D})$. However, $H^\infty_K(\mathbb{D})$ is an algebra, so $z \cdot z = z^2 \in H^\infty_K(\mathbb{D})$, a contradiction. Therefore, $1 < k - 1$. But since $k - 1 < k$ with $k - 1 \notin K$ and $k \in K$, Lemma 3.1.(iv) implies that $k - (k - 1) = 1 \in K$, another contradiction. Thus, we must have $1 \in K$. □

If K only has consecutive integers, then either K is finite, containing all positive integers less than some integer, or $K = \mathbb{Z}_+$.

Proposition 3.9. *Suppose $H^\infty_K(\mathbb{D})$ is an algebra. If K only contains consecutive integers, then $K = \{1, 2, \ldots, k\}$ for some integer k, or $K = \mathbb{Z}_+$.*

Proof. Suppose for each $k \in \mathbb{Z}_+$, $K \neq \{1, 2, \ldots, k\}$. By Proposition 3.8, $1 \in K$. But $K \neq \{1\}$. Since K only has consecutive integers, $2 \in K$. Similarly, if $k \in K$, then $k + 1 \in K$. Thus, $K = \mathbb{Z}_+$. □

Lastly, we show that if there is a smallest integer m not in K, then K contains all positive integers less than m.

Proposition 3.10. *Let $H^\infty_K(\mathbb{D})$ be an algebra with $K \neq \mathbb{Z}_+$. Let m be the smallest positive integer not in K. Then K contains every positive integer $j < m$.*

Proof. By Proposition 3.8, $K \neq \emptyset$. Since K and $\mathbb{Z}_+ \setminus K$ partition $\mathbb{Z}_+$ and $m = \min\left(\mathbb{Z}_+ \setminus K\right)$, every positive integer $j < m$ must be in K. □

We show that a smallest positive integer $m \notin K$ can be found, so that a function f can be written as a product of functions involving m.

Lemma 3.11. *Let $K \subset \mathbb{Z}_+$ be a proper nonempty set such that $H^\infty_K(\mathbb{D})$ is an algebra. Let $f \in H^\infty_K(\mathbb{D})$ with $f(0) = 0$. Then there exists a smallest positive integer $m \notin K$ and an analytic function h on $\mathbb{D}$ such that*

$$f(z) = z^m h(z).$$

Proof. It is enough to assume that f is not identically zero and nonconstant on $\mathbb{D}$. If K only contains consecutive integers, then because K is a proper subset of $\mathbb{Z}_+$, Proposition 3.9 tells us that $K = \{1, 2, 3 \ldots, k\}$, for some $k \in \mathbb{Z}_+$. Then the smallest positive integer not in K is $k + 1$. Now, if $f \in H^\infty_K(\mathbb{D})$ and $f(0) = 0$, it follows that f has the following representation:

$$\begin{aligned} f(z) &= \alpha_{k+1} z^{k+1} + \sum_{i=k+2}^{\infty} \alpha_i z^i \\ &= z^{k+1} h(z), \end{aligned}$$

where $h(z) = \alpha_{k+1} + \sum_{k+2}^{\infty} \alpha_i z^i$ is analytic, as desired.

If K contains a pair of nonconsecutive integers, let k_i and k_{i+1} denote the smallest such pair. Then $m = k_i + 1$ is the smallest positive integer not in K. By Proposition 3.10 every positive integer $j < k_i + 1$ is in K, so that $f^{(j)}(0) = 0$. Because $f(0) = 0$, we know that f has the following representation for all positive integers $\ell \notin K \cup \{m\}$:

$$\begin{aligned} f(z) &= \alpha_m z^m + \sum_{\ell} c_\ell z^\ell \\ &= z^m h(z), \end{aligned}$$

where $h(z) = \alpha_m + \sum_\ell c_\ell z^{\ell-m}$ is an analytic function, as desired. □

Let f be an analytic function on $\mathbb{D}$ such that $||f||_\infty \leq 1$. If there is a point $z_0 \in \mathbb{D}$ such that $f(z_0) \in \mathbb{D}$, then the image $f(\mathbb{D})$ is completely contained in $\mathbb{D}$. To prove this claim, recall that the Open Mapping Theorem says that the image of an open set under a nonconstant analytic mapping is open.

Lemma 3.12. *Let $f \in H^\infty(\mathbb{D})$ such that $||f||_\infty \leq 1$. If there is a point $z_0 \in \mathbb{D}$ such that $f(z_0) \in \mathbb{D}$, then $|f(z)| < 1$ for all $z \in \mathbb{D}$.*

Proof. If f is constant, then $f(z) = f(z_0)$ for all $z \in \mathbb{D}$, and the result follows. Otherwise, the Open Mapping Theorem and $||f||_\infty \leq 1$ imply that the image $f(\mathbb{D})$ is an open set contained in $\mathbb{D}$. Thus, $|f(z)| < 1$ for all $z \in \mathbb{D}$. □

4 The Interpolation Theorem On The Subalgebra $H^\infty_K(\mathbb{D})$

We are now ready to present our main theorem.

Theorem 4.1. *Let $K = \{1, k_1, k_2, \ldots, k_p\}$ with $1 < k_1 < k_2 < \ldots < k_p$ denote a set of positive integers such that $H^\infty_K(\mathbb{D})$ is an algebra. Let $z_1, z_2, \ldots, z_n$ be distinct points in $\mathbb{D}$ and $w_1, w_2, \ldots, w_n$ be points in $\mathbb{D}$.*
There exists an analytic function $f \in H^\infty_K(\mathbb{D})$ with $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$ and $||f||_\infty \leq 1$ if and only if there exists a $\lambda \in \mathbb{D}$ such that

$$\left[\frac{z_i^{m\eta}\,\overline{z_j^{m\eta}} - \varphi_\lambda^\eta(w_i)\,\overline{\varphi_\lambda^\eta(w_j)}}{1 - z_i\overline{z_j}}\right] \quad \textit{is} \quad \textit{positive} \quad \textit{semidefinite},$$

where m is the smallest positive integer not in K, and $\eta = \left\lceil \frac{k_p}{m} \right\rceil$.

Proof. First, we suppose $z_i \neq 0$ for all $i = 1, 2, \ldots, n$. Assume that there exists $f \in H^\infty_K(\mathbb{D})$ such that $||f||_\infty \leq 1$ and $f(z_i) = w_i$. Set $\lambda = f(0)$, which is in $\mathbb{D}$ by Lemma 3.12. Define $g(z) = \varphi_\lambda(f(z))$. Since $\varphi_\lambda \in H^\infty(\mathbb{D})$ and $f \in H^\infty_K(\mathbb{D})$, it follows from Lemma 3.6 that $g \in H^\infty_K(\mathbb{D})$. Moreover, $g(z_i) = \varphi_\lambda(w_i)$ for $i = 1, 2 \ldots, n$, and $g(0) = \varphi_\lambda(\lambda) = 0$.

Consequently, we know from Lemma 3.11 that a smallest positive integer $m \notin K$ exists such that $g(z) = z^m \ell(z)$ where $||\ell||_\infty \leq 1$. Since $H^\infty_K(\mathbb{D})$ is an algebra, we have that

$$\underbrace{g(z)\,g(z)\,\cdots\,g(z)}_{\eta = \left\lceil \frac{k_p}{m} \right\rceil \text{ times}} = g^\eta(z) = \varphi_\lambda^\eta(f(z)) = z^{m\eta} h(z) \in H^\infty_K(\mathbb{D}),$$

where $h(z) = \ell^\eta(z)$ and $||h||_\infty \leq 1$. Applying the Nevanlinna-Pick Theorem on $h \in H^\infty(\mathbb{D})$ and the points $z_1, z_2, \ldots, z_n$, the matrix

$$\left[\frac{1 - h(z_i)\overline{h(z_j)}}{1 - z_i\overline{z_j}}\right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

Moreover, the matrix

$$\left[\frac{z_i^{m\eta}\overline{z_j^{m\eta}} - g^\eta(z_i)\,\overline{g^\eta(z_j)}}{1 - z_i\overline{z_j}}\right]$$

equals

$$\begin{bmatrix} z_1^{m\eta} & 0 & \ldots & 0 \\ 0 & z_2^{m\eta} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & z_n^{m\eta} \end{bmatrix} \left[\frac{1 - h(z_i)\overline{h(z_j)}}{1 - z_i\overline{z_j}}\right] \begin{bmatrix} \overline{z_1^{m\eta}} & 0 & \ldots & 0 \\ 0 & \overline{z_2^{m\eta}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \overline{z_n^{m\eta}} \end{bmatrix}.$$

Consequently,

$$\left[\frac{z_i^{m\eta}\overline{z_j^{m\eta}} - g^\eta(z_i)\,\overline{g^\eta(z_j)}}{1 - z_i\overline{z_j}}\right] = \left[\frac{z_i^{m\eta}\overline{z_j^{m\eta}} - \varphi_\lambda^\eta(w_i)\,\overline{\varphi_\lambda^\eta(w_j)}}{1 - z_i\overline{z_j}}\right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

Now suppose $z_1 = 0$. Then $\lambda = f(0) = f(z_1) = w_1$, and thus the first row and column of the matrix above are zero; and the same reasoning applies to the remaining entries of the matrix.

Conversely, it suffices to assume that $z_i \neq 0$ for all $i = 1, 2, \ldots, n$. Suppose there exists a $\lambda \in \mathbb{D}$ such that

$$\left[\frac{z_i^{m\eta}\overline{z_j^{m\eta}} - \varphi_\lambda^\eta(w_i)\,\overline{\varphi_\lambda^\eta(w_j)}}{1 - z_i\overline{z_j}}\right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

Since the above matrix equals

$$\begin{bmatrix} z_1^{m\eta} & 0 & \dots & 0 \\ 0 & z_2^{m\eta} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_n^{m\eta} \end{bmatrix} \left[\frac{1 - z_i^{-m\eta} \varphi_\lambda^\eta(w_i)\, \overline{z_j^{-m\eta}\, \varphi_\lambda^\eta(w_j)}}{1 - z_i \overline{z_j}} \right] \begin{bmatrix} \overline{z_1^{m\eta}} & 0 & \dots & 0 \\ 0 & \overline{z_2^{m\eta}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{z_n^{m\eta}} \end{bmatrix},$$

we get that

$$\left[\frac{1 - z_i^{-m\eta} \varphi_\lambda^\eta(w_i)\, \overline{z_j^{-m\eta}\, \varphi_\lambda^\eta(w_j)}}{1 - z_i \overline{z_j}} \right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

By the Nevanlinna-Pick Theorem, there exists an $h \in H^\infty(\mathbb{D})$ with $||h||_\infty \leq 1$, so $h(z_i) = z_i^{-m\eta} \varphi_\lambda^\eta(w_i)$. Define $f(z) = \psi(z^{m\eta} h(z))$ where ψ is the inverse Möbius transformation of φ_λ^η. Clearly, $f \in H^\infty(\mathbb{D})$, $||f||_\infty \leq 1$, and $f(z_i) = \psi(z_i^{m\eta}\, h(z_i)) = \psi(z_i^{m\eta}\, z_i^{-m\eta}\, \varphi_\lambda^\eta(w_i)) = w_i$, for all $i = 1, 2, \dots, n$.

Moreover, $z^{m\eta} h(z) \in H^\infty(\mathbb{D})$. Now, for each $k \in K$, we have that

$$\big(z^{m\eta} h(z)\big)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \big(z^{m\eta}\big)^{(i)} \big(h(z)\big)^{(k-i)}.$$

And, for each $i = 0, 1, \dots, k$,

$$\big(z^{m\eta}\big)^{(i)} = \frac{k!}{(k-i)!} \cdot z^{m\eta - i}$$

By Lemma 3.7, $m\eta - i > 0$, so $\big(z^{m\eta} h(z)\big)^{(k)} = 0$ at $z = 0$. Thus, $z^{m\eta} h(z) \in H_K^\infty(\mathbb{D})$. Since $\psi \in H^\infty(\mathbb{D})$, we conclude from Lemma 3.6 that $f(z) = \psi(z^{m\eta} h(z)) \in H_K^\infty(\mathbb{D})$. □

Remark. If $K = \{1\}$, then $m = 2$ and $\eta = 1$ give us the result of [9] on $H_{\{1\}}^\infty(\mathbb{D})$.

Corollary 4.2. *Let K be an infinite proper subset of $\mathbb{Z}_+$ such that $H_K^\infty(\mathbb{D})$ is an algebra and*

$$\mathbb{Z}_+ \setminus K = \{n_1 d, n_2 d, \dots, n_p d, N_0 d, (N_0 + j)d : j \in \mathbb{Z}_+\}$$

for some positive integer d, some finite set $\{n_i\}_{i=1}^p \subset \mathbb{Z}_+$ where $n_1 < n_2 < \cdots < n_p$ and a positive integer $N_0 > n_p$. Moreover, let $K_1 \subset \mathbb{Z}_+$ such that

$$\mathbb{Z}_+ \setminus K_1 = \{n_1, n_2, \dots, n_p, N_0, N_0 + j \ : \ j \in \mathbb{Z}_+\}.$$

Let $z_1, z_2, \dots, z_n$ be distinct points in $\mathbb{D}$ and $w_1, w_2, \dots, w_n$ be points in $\mathbb{D}$. Then there exists an $f \in H_K^\infty(\mathbb{D})$ with $f(z_i) = w_i$ for $i = 1, 2, \dots, n$ and $||f||_\infty \leq 1$ if and only if there exists a $\lambda \in \mathbb{D}$ such that

$$\left[\frac{z_i^{n_1 d\, \eta} \overline{z_j^{n_1 d\, \eta}} - \varphi_\lambda^\eta(w_i) \overline{\varphi_\lambda^\eta(w_j)}}{1 - z_i^d\, \overline{z_j^d}} \right] \quad \textit{is} \quad \textit{positive} \quad \textit{semidefinite}$$

where n_1 is the smallest positive integer not in K_1 and $\eta = \left\lceil \frac{N_0 - 1}{n_1} \right\rceil$.

Proof. Let $f \in H_K^\infty(\mathbb{D})$ such that $f(z_i) = w_i$ and $||f||_\infty \leq 1$. By our assumption on $\mathbb{Z}_+ \setminus K$, we see that f has the following representation

$$f(z) = \alpha_0 + \alpha_1\, z^{n_1 d} + \alpha_2\, z^{n_2 d} + \cdots + \alpha_p\, z^{n_p d} + \alpha_{p+1}\, z^{N_0 d} + \sum_{j=1}^{\infty} \alpha_{p+j+1}\, z^{(N_0 + j)d},$$

where $\alpha_i \in \mathbb{C}$. Setting $V = z^d$, we have a function F defined by

$$F(V) = \alpha_0 + \alpha_1\, V^{n_1} + \alpha_2\, V^{n_2} + \cdots + \alpha_p\, V^{n_p} + \alpha_{p+1}\, V^{N_0} + \sum_{j=1}^{\infty} \alpha_{p+j+1}\, V^{N_0 + j}.$$

Consequently, F is contained in the algebra $H_{K_1}^\infty(\mathbb{D})$. Also, $||F||_\infty \leq 1$ since $||f||_\infty \leq 1$, and $F(V_i) = F(z_i^d) = f(z_i) = w_i$.

By Theorem 4.1, there exists a $\lambda \in \mathbb{D}$ such that

$$\left[\frac{V_i^{n_1\eta}\,\overline{V_j^{n_1\eta}} - \varphi_\lambda^\eta(V_i)\overline{\varphi_\lambda^\eta(V_j)}}{1 - V_i\,\overline{V_j}}\right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite}$$

where n_1 is the smallest positive integer not in K_1 and $\eta = \left\lceil \frac{N_0-1}{n_1} \right\rceil$. Since $V_i = z_i^d$ and $\overline{V_j} = \overline{z_j^d}$, we conclude that

$$\left[\frac{z_i^{n_1 d\,\eta}\overline{z_j^{n_1 d\,\eta}} - \varphi_\lambda^\eta(w_i)\overline{\varphi_\lambda^\eta(w_j))}}{1 - z_i^d\,\overline{z_j^d}}\right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

Conversely, suppose there is a $\lambda \in \mathbb{D}$ such that the above matrix is positive semidefinite. Set $V_i = z_i^d$ and $\overline{V_j} = \overline{z_j^d}$, so that

$$\left[\frac{V_i^{n_1\eta}\,\overline{V_j^{n_1\eta}} - \varphi_\lambda^\eta(w_i)\overline{\varphi_\lambda^\eta(w_j)}}{1 - V_i\,\overline{V_j}}\right] \quad \text{is} \quad \text{positive} \quad \text{semidefinite.}$$

By Theorem 4.1, there exists an $F \in H^\infty_{K_1}(\mathbb{D})$ such that $F(V_i) = w_i$ and $||F||_\infty \leq 1$. Further, we have

$$F(V) = \alpha_0 + \alpha_1\,V^{n_1} + \alpha_2\,V^{n_2} + \cdots + \alpha_p\,V^{n_p} + \alpha_{p+1}\,V^{N_0} + \sum_{j=1}^{\infty}\alpha_{p+j+1}\,V^{N_0+j}.$$

Defining $f(z) = F(z^d)$, we see that

$$f(z) = \alpha_0 + \alpha_1\,z^{n_1 d} + \alpha_2\,z^{n_2 d} + \cdots + \alpha_p\,z^{n_p d} + \alpha_{p+1}\,z^{N_0 d} + \sum_{j=1}^{\infty}\alpha_{p+j+1}\,z^{(N_0+j)d}.$$

Clearly, $f \in H^\infty_K(\mathbb{D})$, $||f||_\infty \leq 1$ and $f(z_i) = F(z_i^d) = w_i$, for all $i = 1, 2, \ldots, n$. We conclude that f is our desired interpolation function. □

5 Conclusion and Open Problems

In this paper, we have extended the interpolation result for $H^\infty_{\{1\}}(\mathbb{D})$ to all algebras $H^\infty_K(\mathbb{D})$ where K is a nonempty subset of $\mathbb{Z}_+$. However, questions have naturally arisen during this investigation. For the classical Nevanlinna-Pick interpolation result, the interpolation function is unique if and only if the associated Pick matrix is singular. This fact inspires the following question.

Question 5.1. Are there necessary and sufficient conditions to guarantee that the the interpolation function in $H^\infty_K(\mathbb{D})$ is unique?

Lemma 3.6 shows that $H^\infty_K(\mathbb{D})$ is a right ideal in $H^\infty(\mathbb{D})$ for any set K yielding an algebra. However, Example 3.4 suggests that it is not always a left ideal, which leads us to our next question.

Question 5.2. Suppose $g \in H^\infty_K(\mathbb{D})$ and $f \in H^\infty(\mathbb{D})$. Under what conditions is the composition $g \circ f \in H^\infty_K(\mathbb{D})$?

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